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Two-dimensional Coulomb Systems in a Disk with Ideal Dielectric Boundaries

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We study two-dimensional Coulomb systems confined in a disk with ideal dielectric boundaries. In particular we consider the two-component plasma in detail. When the coulombic coupling constant $\Gamma = 2$ the model is exactly solvable. We compute the grand potential, densities and correlations. We show that the grand potential has a universal logarithmic finite-size correction as predicted in previous works. This logarithmic finite-size correction is also found in the free energy of another solvable model: the one-component plasma.

KEY WORDS: Coulomb systems; solvable models; Neumann boundary conditions; finite-size effects; universality; correlations.

1. INTRODUCTION

Solvable models of Coulomb systems have attracted attention for quite some time. Very recently the two-dimensional two-component plasma, a model of two species of point-particles with opposite charges $\pm q$ at inverse temperature $\beta = 1/k_B T$, has been solved in its whole range of stability $\Gamma := \beta q^2 < 2$ by using a mapping of this system onto a sine-Gordon field theory.[(1)] With this mapping the grand-partition function and other bulk properties of the system can be computed exactly. Also some surface properties near a metallic wall[(2)] and an ideal dielectric wall[(3)] have been investigated. However this mapping onto a sine-Gordon model does not give (yet) any information on correlation functions.

When the coupling constant $\Gamma = 2$ the corresponding sine-Gordon model is at its free fermion point and additional information on the system can be obtained. This fact is well known and much work has been done on

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the two-component plasma at $\Gamma = 2$ since the pioneer work of Gaudin on this model.[4] In particular the two-component plasma at $\Gamma = 2$ has been studied in different geometries: near a plane hard wall or a polarizable interface,[5] a metallic wall,[6] in a disk with hard walls[7] and in a disk with metallic walls[8]....

However it was not until very recently that the special case of ideal dielectric boundaries (that is Neumann boundary conditions imposed on the electric potential) has been studied by Jancovici and Šamaj[9] for a system near an infinite plane wall or confined in a strip. The technical difficulty with this kind of boundary conditions is that the two-component plasma must be mapped onto a four-component free Fermi field instead of a two-component free Fermi field as in all other cases of boundary conditions.

A very interesting result of ref. 9 is that when the system is confined in a strip of width W made of ideal dielectric walls, the grand potential per unit length exhibits a universal finite-size correction equal to $\pi/24W$ which is the same finite-size correction for a system confined in a strip made of ideal conductor walls.[8] These finite-size corrections have been explained[8] by noting that if one disregards microscopic detail the grand-partition function of a Coulomb system is the inverse of the partition function of the Gaussian model.

Due to this analogy with the Gaussian field theory, a Coulomb system in a confined geometry is expected to exhibit universal logarithmic finite size-corrections, for instance in a disk of radius R the grand potential should have a correction $(1/6)\ln R$. For the analogy with the Gaussian field theory to be complete one should impose conformally invariant boundary conditions to the electric potential, for instance Dirichlet boundary conditions (ideal metallic walls) or Neumann boundary conditions (ideal dielectric walls). The case of Dirichlet boundary conditions was treated in ref. 8.

In this paper we study the two-component plasma at $\Gamma = 2$ in a disk with Neumann boundary conditions. One of the main motivations for this work is to show that the system exhibits in fact the expected universal logarithmic finite-size correction.

The outline of this paper is the following. In Section 2, we present the model and adapting the method of ref. 9 we map the two-component plasma onto a four-component free Fermi field. In Section 3, we compute the grand potential and its large-R expansion. We also compute the large-R expansion of the free energy of the one-component plasma which was solved some time ago.[11] In Section 4, we compute densities and correlation functions.

2. THE MODEL

The system is composed of two species of point-particles with opposite charges $\pm q$. The particles are confined in a disk D of radius R. It will be very useful to work with the complex coordinate $z = re^{i\phi}$ of a point $\mathbf{r}$. The material outside the disk is supposed to be an ideal dielectric. This imposes Neumann boundary conditions on the electric potential: the interaction potential $v(\mathbf{r}, \mathbf{r}')$ between two charges located at $\mathbf{r}$ and $\mathbf{r}'$ is the solution of Poisson equation

$$\Delta_{\mathbf{r}} v(\mathbf{r}, \mathbf{r}') = -2\pi\delta(\mathbf{r}-\mathbf{r}') \tag{2.1}$$

with Neumann boundary conditions. However, it is well-known[10] that any solution of Poisson equation (2.1) in a closed domain D cannot satisfy homogeneous Neumann boundary conditions $\partial_n v(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in \partial D$, since Gauss theorem implies that $\oint_{\partial D} \partial_n v(\mathbf{r}, \mathbf{r}')\, dl = -2\pi$. A natural choice is to impose inhomogeneous Neumann boundary conditions to the potential $\partial_n v(\mathbf{r}, \mathbf{r}') = -2\pi/L$, with L the perimeter of the domain D. In the case of the disk of radius R this gives

$$\partial_n v(\mathbf{r}, \mathbf{r}') = -1/R \qquad \text{for} \quad \mathbf{r} \in \partial D \tag{2.2}$$

This impossibility for the electric potential between pairs to obey homogeneous Neumann boundary conditions is not a problem for a system globally neutral in which the total electric potential will satisfy homogeneous Neumann boundary conditions if the pair potential satisfies (2.2). It should be noted that for an infinite wall boundary it is possible for the electric potential between pairs to obey homogeneous Neumann boundary conditions.[9]

The solution of Poisson equation (2.1) with boundary conditions (2.2) in a disk can be obtained obtained by usual methods of electrostatics (images, etc ...). The solution is

$$v(\mathbf{r}, \mathbf{r}') = -\ln \frac{|z-z'|\,|R^2 - z\bar{z}'|}{a^2 R} \tag{2.3}$$

where a is some irrelevant length scale and $\bar{z}$ is the complex conjugate of z. It can be easily checked that solution (2.3) can also be obtained as the limit of the boundary value problem where outside the disk there is a dielectric with dielectric constant $\epsilon \to 0$ (up to some constant terms).[11]

It will be useful to introduce the image $\mathbf{r}^*$ of a point $\mathbf{r}$ by $z^* = R^2/\bar{z}$. The electric potential between pairs can then be written as

$$v(\mathbf{r}, \mathbf{r}') = -\ln \frac{|z-z'|\,|z'^*-z|\,|\bar{z}'|}{a^2 R} \tag{2.4}$$

and can be interpreted as the potential in $\mathbf{r}$ due to a charge in $\mathbf{r}'$ and a image charge with equal sign and magnitude located at $\mathbf{r}'^*$.

If the temperature is not high enough the two-dimensional two-component plasma is not well defined, this is due to the collapse between pairs of opposite sign. If the coupling constant $\Gamma := \beta q^2 < 2$ the system is stable. Here we will study the case $\Gamma = 2$. In order to avoid the collapse we will work initially with two interwoven lattices U and V. Positive particles with complex coordinates $\{u_i\}$ live in the sites of lattice U and negative particles with complex coordinates $\{v_i\}$ live in the sites of V. We shall work in the grand-canonical ensemble and will only consider neutral configurations. The Boltzmann factor of a system with N positive particles and N negatives particles at $\Gamma = 2$ is

$$e^{-\beta H_N} = a^{2N} \prod_{i=1}^{N} (R^2 - |u_i|^2)(R^2 - |v_i|^2) \times \frac{\prod_{i<j} (|u_i - u_j|\,|v_i - v_j|\,|R^2 - u_i\bar{u}_j|\,|R^2 - v_i\bar{v}_j|)^2}{\prod_{i,j} (|u_i - v_j|\,|R^2 - u_i\bar{v}_j|)^2} \tag{2.5}$$

The first product corresponds to the self-energies of the particles and the other terms to the interactions between pairs. Introducing the images this can be rewritten as

$$e^{-\beta H_N} = a^{2N} \prod_i \left(\frac{R^2}{\bar{u}_i\bar{v}_i}\right) \prod_{i=1}^{N} (u_i - u_i^*)(v_i - v_i^*) \times \frac{\prod_{i<j} (u_i-u_j)(u_i^*-u_j^*)(u_i-u_j^*)(u_i^*-u_j)(v_i-v_j)(v_i^*-v_j^*)(v_i-v_j^*)(v_i^*-v_j)}{\prod_{i,j} (u_i-v_j)(u_i^*-v_j^*)(u_i-v_j^*)(u_i^*-v_j)} \tag{2.6}$$

By using Cauchy's double alternant formula

$$\det\left(\frac{1}{z_i - z'_j}\right)_{(i,j)\in\{1,\ldots,2N\}^2} = (-1)^{N(2N-1)} \frac{\prod_{i<j} (z_i - z_j)(z'_i - z'_j)}{\prod_{i,j} (z_i - z'_j)} \tag{2.7}$$

with

$$z_{2i-1} = u_i, \qquad z_{2i} = u_i^*, \qquad z'_{2i-1} = v_i, \qquad z'_{2i} = v_i^*, \tag{2.8}$$

we find that the Boltzmann factor can be written as a $2N \times 2N$ determinant

$$e^{-\beta H_N} = (-1)^N \prod_i \left(\frac{a^2R^2}{\bar{u}_i\bar{v}_i}\right) \det \begin{pmatrix} \dfrac{1}{u_i - v_j} & \dfrac{1}{u_i - v_j^*} \\ \dfrac{1}{u_i^* - v_j} & \dfrac{1}{u_i^* - v_j^*} \end{pmatrix} \tag{2.9}$$

Introducing the factors $Ra/\bar{u}_i$ into the last N rows of the determinant and the factors $Ra/\bar{v}_i$ into the last N columns, we finally arrive at the expression

$$e^{-\beta H_N} = (-1)^N \det \begin{pmatrix} \dfrac{a}{u_i - v_j} & \dfrac{aR}{u_i\bar{v}_j - R^2} \\ \dfrac{aR}{R^2 - \bar{u}_i v_j} & \dfrac{a}{\bar{v}_j - \bar{u}_i} \end{pmatrix} \tag{2.10}$$

The grand-canonical partition function with position dependent fugacities $\zeta(u)$ and $\zeta(v)$ for positive and negative particles reads

$$\Xi = 1 + \sum_{N=1}^{\infty} \sum_{\substack{u_1 < \cdots < u_N \in U \\ v_1 < \cdots < v_N \in V}} \left(\prod_{i=1}^{N} \zeta(u_i)\zeta(v_i)\right) e^{-\beta H_N} \tag{2.11}$$

We shall now closely follow ref. 9 to show that the grand-partition function can be written as a ratio of two Pfaffians. Using the same notations as in ref. 9, let us introduce a couple of Grassmann anticommuting variables (ψ_u^1, ψ_u^2) for each site $u \in U$ and similar Grassmann variables for each site in V. The Grassmann integral for any antisymmetric matrix $\mathbf{A}$

$$Z_0 = \int d\theta \exp\left(\tfrac{1}{2}\, {}^t\theta \mathbf{A} \theta\right) \tag{2.12}$$

with ${}^t\theta = (..., \psi_u^1, \psi_u^2, ..., \psi_v^1, \psi_v^2, ...)$ and $d\theta = \prod_v d\psi_v^2 \, d\psi_v^1 \prod_u d\psi_u^2 \, d\psi_u^1$ is the Pfaffian of the matrix $\mathbf{A}$

$$Z_0 = \mathrm{Pf}(\mathbf{A}) \tag{2.13}$$

Let us denote the average of a quantity $\mathcal{O}$ by

$$\langle \mathcal{O} \rangle = \frac{1}{Z_0} \int d\theta \, \mathcal{O} \exp\left(\frac{1}{2}\, {}^t\theta \mathbf{A} \theta\right) \tag{2.14}$$

It is well known that[12]

$$\langle \theta_i \theta_j \rangle = (A^{-1})_{ji} \tag{2.15}$$

For our purposes let us define the matrix **A** as having inverse elements

$$(A^{-1})^{\alpha\beta}_{uu'} = 0 \tag{2.16a}$$

$$(A^{-1})^{\alpha\beta}_{vv'} = 0 \tag{2.16b}$$

$$(A^{-1})^{\alpha\beta}_{uv} = \begin{pmatrix} \dfrac{a}{u-v} & \dfrac{aR}{u\bar{v}-R^2} \\ \dfrac{aR}{R^2-\bar{u}v} & \dfrac{a}{\bar{v}-\bar{u}} \end{pmatrix} \tag{2.16c}$$

$$(A^{-1})^{\alpha\beta}_{vu} = \begin{pmatrix} \dfrac{a}{v-u} & \dfrac{aR}{v\bar{u}-R^2} \\ \dfrac{aR}{R^2-\bar{v}u} & \dfrac{a}{\bar{u}-\bar{v}} \end{pmatrix} \tag{2.16d}$$

The indexes $(\alpha, \beta) \in \{1, 2\}^2$ label the rows and columns respectively. The matrix **A** is clearly antisymmetric as required. We now introduce the antisymmetric matrix **M**

$$M^{\alpha\beta}_{uu'} = \delta_{uu'} \begin{pmatrix} 0 & \zeta(u) \\ -\zeta(u) & 0 \end{pmatrix} \tag{2.17a}$$

$$M^{\alpha\beta}_{vv'} = \delta_{vv'} \begin{pmatrix} 0 & \zeta(v) \\ -\zeta(v) & 0 \end{pmatrix} \tag{2.17b}$$

$$M^{\alpha\beta}_{uv} = 0 \tag{2.17c}$$

$$M^{\alpha\beta}_{vu} = 0 \tag{2.17d}$$

The Grassmann integral

$$Z = \int d\theta \exp\left[\tfrac{1}{2}\, {}^t\theta(\mathbf{A}+\mathbf{M})\,\theta\right] \tag{2.18}$$

is equal to

$$Z = \mathrm{Pf}(\mathbf{A}+\mathbf{M}) \tag{2.19}$$

The ratio Z/Z_0 can be expanded in powers of the fugacities as

$$\frac{Z}{Z_0}=1+\sum_{N=1}^{\infty}\sum_{\substack{u_1<\cdots<u_N\in U\\ v_1<\cdots<v_N\in V}}\left(\prod_{i=1}^{N}\zeta(u_i)\zeta(v_i)\right)\left\langle\prod_{i=1}^{N}(\psi_{u_i}^1\psi_{u_i}^2\psi_{v_i}^1\psi_{v_i}^2)\right\rangle \tag{2.20}$$

Using Wick theorem for fermions we find that

$$\left\langle\prod_{i=1}^{N}(\psi_{u_i}^1\psi_{u_i}^2\psi_{v_i}^1\psi_{v_i}^2)\right\rangle=(-1)^N\det\begin{pmatrix}\dfrac{a}{u_i-v_j} & \dfrac{aR}{u_i\bar{v}_j-R^2}\\[2ex] \dfrac{aR}{R^2-\bar{u}_iv_j} & \dfrac{a}{\bar{v}_j-\bar{u}_i}\end{pmatrix} \tag{2.21}$$

Comparing equations (2.20) and (2.21) with equations (2.11) and (2.10) we immediately conclude that the grand-canonical partition of the Coulomb system Ξ is the ratio of two Pfaffians

$$\Xi=\frac{Z}{Z_0}=\frac{\mathrm{Pf}(\mathbf{A}+\mathbf{M})}{\mathrm{Pf}(\mathbf{A})} \tag{2.22}$$

Using the fact that the Pfaffian is the square root of the determinant we can write the grand potential as

$$\beta\Omega=-\ln\Xi=-\tfrac{1}{2}\ln\det(1+\mathbf{K})=-\tfrac{1}{2}\operatorname{Tr}\ln(1+\mathbf{K}) \tag{2.23}$$

where the matrix $\mathbf{K}$ is $\mathbf{K}=\mathbf{M}\mathbf{A}^{-1}$ and has matrix elements

$$K_{ss'}^{\alpha\beta}(\mathbf{r},\mathbf{r}')=\delta_{s,\,-s'}\zeta_s(\mathbf{r})\begin{pmatrix}\dfrac{aR}{R^2-\bar{z}z'} & \dfrac{a}{\bar{z}'-\bar{z}}\\[2ex] \dfrac{a}{z'-z} & \dfrac{aR}{R^2-z\bar{z}'}\end{pmatrix} \tag{2.24}$$

We have introduced the notation $u=(\mathbf{r},+)$, $v=(\mathbf{r},\,-)$, $\zeta(u)=\zeta_+(\mathbf{r})$, $\zeta(v)=\zeta_-(\mathbf{r})$ and $(s,\,s')\in\{-,+\}^2$.

3. THE GRAND POTENTIAL

3.1. Formal expression of the grand potential

To compute explicitly the grand potential from equation (2.23) we must find the eigenvalues of $\mathbf{K}$. From now on we will consider the continuum limit where the spacing of the lattices U and V goes to zero. In this limit it is natural to work with the rescaled fugacity[5] $m = 2\pi a\zeta/S$ where S is the area of a lattice cell. Also in this limit some bulk quantities will be divergent, because of the collapse of particle of opposite sign, and must be cutoff. Correlations in contrast have a well defined continuum limit.

Let $\{\psi_s^{(\alpha)}(\mathbf{r})\}_{s=\pm;\alpha=1,2}$ be the eigenvectors of $m^{-1}\mathbf{K}$ and $1/\lambda$ the corresponding eigenvalues. The eigenvalue problem for $\mathbf{K}$ is the set of integral equations

$$\frac{\lambda}{2\pi}\int_D d\mathbf{r}'\left[\frac{R}{R^2-\bar{z}z'}\psi_{-s}^{(1)}(\mathbf{r}')+\frac{1}{\bar{z}'-\bar{z}}\psi_{-s}^{(2)}(\mathbf{r}')\right]=\psi_s^{(1)}(\mathbf{r}) \tag{3.1a}$$

$$\frac{\lambda}{2\pi}\int_D d\mathbf{r}'\left[\frac{1}{z'-z}\psi_{-s}^{(1)}(\mathbf{r}')+\frac{R}{R^2-z\bar{z}'}\psi_{-s}^{(2)}(\mathbf{r}')\right]=\psi_s^{(2)}(\mathbf{r}) \tag{3.1b}$$

These integral equations (3.1) can be transformed into differential equations plus some boundary conditions using the well-known identities

$$\frac{\partial}{\partial z}\frac{1}{\bar{z}-\bar{z}'}=\frac{\partial}{\partial \bar{z}}\frac{1}{z-z'}=\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{3.2}$$

Applying ∂_z to equation (3.1a) and $\partial_{\bar{z}}$ to equation (3.1b) yields

$$-\frac{\lambda}{2}\psi_{-s}^{(2)}(\mathbf{r})=\partial_z\psi_s^{(1)}(\mathbf{r}) \tag{3.3a}$$

$$-\frac{\lambda}{2}\psi_{-s}^{(1)}(\mathbf{r})=\partial_{\bar{z}}\psi_s^{(2)}(\mathbf{r}) \tag{3.3b}$$

These differential equations (3.3) can be combined into the Laplacian eigenvalue problem

$$\Delta\psi_s^{(\alpha)}=\lambda^2\psi_s^{(\alpha)} \tag{3.4}$$

which must be complemented with the following boundary conditions. If $\mathbf{r}=\mathbf{R}$ is on the boundary, $z=Re^{i\phi}$, it can be easily seen from integral equations (3.1) that

$$\psi_s^{(1)}(\mathbf{R})+e^{i\phi}\psi_s^{(2)}(\mathbf{R})=0 \tag{3.5}$$

An elementary solution of equation (3.4) in the present disk geometry is

$$\psi_s^{(2)}(\mathbf{r})=A_s e^{i\ell\phi}I_\ell(\lambda r) \tag{3.6}$$

with, from equation (3.3b),

$$\psi_{-s}^{(1)}(\mathbf{r}) = -A_s e^{i(\ell+1)\phi} I_{\ell+1}(\lambda r) \tag{3.7}$$

where I_ℓ is a modified Bessel functions of order ℓ. Boundary conditions (3.5) yield the following homogeneous linear system for the coefficients A_s

$$-A_- I_{\ell+1}(\lambda R) + A_+ I_\ell(\lambda R) = 0 \tag{3.8a}$$

$$A_- I_\ell(\lambda R) - A_+ I_{\ell+1}(\lambda R) = 0 \tag{3.8b}$$

For this system to have non trivial solutions its determinant must vanish. This gives the equation for the eigenvalue λ

$$I_{\ell+1}(\lambda R)^2 - I_\ell(\lambda R)^2 = 0 \tag{3.9}$$

From equation (2.23) the grand potential then reads

$$\beta\Omega = -\frac{1}{2}\sum_{\ell=-\infty}^{+\infty} \ln \prod_\lambda \left(1+\frac{m}{\lambda}\right) = -\sum_{\ell=0}^{+\infty} \ln \prod_\lambda \left(1+\frac{m}{\lambda}\right) \tag{3.10}$$

where the product runs for all λ solution of equation (3.9). The last equality in equation (3.10) comes from noticing that a change $\ell \to -\ell - 1$ leave equation (3.9) invariant. To evaluate the product in equation (3.10), let us introduce the analytic function for ℓ positive

$$f_\ell(z) = (I_\ell(zR)^2 - I_{\ell+1}(zR)^2)\left(\left(\frac{2}{zR}\right)^\ell \ell!\right)^2 \tag{3.11}$$

The zeros of this function are the eigenvalues λ and it can be checked that $f'_\ell(z)/(z f_\ell(z)) \to 0$ as $z \to \infty$, so this function can be factorized as a Weierstrass product

$$f_\ell(z) = f_\ell(0)\, e^{f'_\ell(0)\, z/f_\ell(0)} \prod_\lambda \left(1-\frac{z}{\lambda}\right) e^{z/\lambda} \tag{3.12}$$

This function satisfies $f_\ell(0) = 1$, $f'_\ell(0) = 0$, and $f_\ell(z) = f_\ell(-z)$ so its zeros come in pairs of opposite sign and as a consequence the exponential factors in Weierstrass product (3.12) cancel out. We finally have

$$f_\ell(z) = \prod_\lambda (1 - z/\lambda) \tag{3.13}$$

where the product runs over all λ solution of equation (3.9).

Then the grand potential can finally be written as

$$\beta\Omega = -\sum_{l=0}^{+\infty} \ln f_\ell(-m)$$
$$= -\sum_{l=0}^{+\infty} \ln\left[\left(\frac{2}{mR}\right)^{2\ell} (\ell!)^2 \left(I_\ell(mR)^2 - I_{\ell+1}(mR)^2\right)\right] \tag{3.14}$$

The above expression is divergent and must be cutoff to a $\ell_{\max} = R/\sigma$ where σ is the ratio of the particles.[5]

It is interesting to notice that the grand potential can be written as the sum of two terms

$$\Omega = \tfrac{1}{2}\left[\Omega_{\substack{\text{ideal}\\\text{cond}}}(m) + \Omega_{\substack{\text{ideal}\\\text{cond}}}(-m)\right] \tag{3.15}$$

where

$$\beta\Omega_{\substack{\text{ideal}\\\text{cond}}}(m) = -2\sum_{l=0}^{\infty} \ln\left[\left(\frac{2}{mR}\right)^{\ell} \ell!\left(I_\ell(mR) + I_{\ell+1}(mR)\right)\right] \tag{3.16}$$

is the grand potential of a two-component plasma at $\Gamma = 2$ confined in a disk with ideal *conductor* boundaries[8] and

$$\beta\Omega_{\substack{\text{ideal}\\\text{cond}}}(-m) = -2\sum_{l=0}^{\infty} \ln\left[\left(\frac{2}{mR}\right)^{\ell} \ell!\left(I_\ell(mR) - I_{\ell+1}(mR)\right)\right] \tag{3.17}$$

is formally the grand potential of the two-component plasma with ideal conductor boundaries but with the sign of the fugacity changed (which of course does not correspond to any physical system). This interesting decomposition of the grand potential also exist in the strip geometry.[9]

3.2. Finite-size corrections

We now compute the large-R expansion of the grand potential (3.14). First using decomposition (3.15) we can use the results of ref. 8 for the expansion of the grand potential with ideal conductor boundaries

$$\beta\Omega_{\substack{\text{ideal}\\\text{cond}}}(m) = -\beta p_b \pi R^2 + \beta\gamma_c 2\pi R + \tfrac{1}{6}\ln(mR) + O(1) \tag{3.18}$$

where the bulk pressure is

$$\beta p_b = \frac{m^2}{2\pi}\left(\ln\frac{2}{m\sigma}+1\right) \tag{3.19}$$

and the surface contribution with ideal conductor boundaries is

$$\beta\gamma_c = m\left(-\frac{1}{2\pi}\ln\frac{2}{m\sigma}-\frac{1}{2\pi}+\frac{1}{4}\right) \tag{3.20}$$

The second contribution to the grand potential can be written as

$$\frac{1}{2}\,\beta\Omega_{\substack{\text{ideal}\\ \text{cond}}}(-m) = -\sum_{\ell=0}^{R/\sigma}\ln\left[1-\frac{I_{\ell+1}(mR)}{I_\ell(mR)}\right]+\frac{1}{2}\,\beta\Omega_{\text{hard}} \tag{3.21}$$

where

$$\beta\Omega_{\text{hard}} = -2\sum_{\ell=0}^{R/\sigma}\ln\left[\ell!\left(\frac{2}{mR}\right)^{\ell} I_\ell(mR)\right] \tag{3.22}$$

is the grand potential of a two-component plasma in a disk with hard wall boundaries.[7] The asymptotic expansion of this term was computed in ref. 7 and reads

$$\beta\Omega_{\text{hard}} = -\beta p_b\pi R^2+\beta\gamma_h 2\pi R+\tfrac{1}{6}\ln(mR)+O(1) \tag{3.23}$$

with the surface contribution for hard walls

$$\beta\gamma_h = m\left(\frac{1}{4}-\frac{1}{2\pi}\right) \tag{3.24}$$

Finally, we only need to compute the asymptotic expansion of

$$S = -\sum_{\ell=0}^{R/\sigma}\ln\left[1-\frac{I_{\ell+1}(mR)}{I_\ell(mR)}\right] \tag{3.25}$$

This can be done with Debye asymptotic expansions for the Bessel functions. First let us write S as

$$S = -\sum_{\ell=0}^{R/\sigma}\ln\left[1-\frac{I'_\ell(mR)}{I_\ell(mR)}+\frac{\ell}{mR}\right] \tag{3.26}$$

The Debye asymptotic expansion for $\ln I_\ell$ is[7, 13]

$$\ln I_\ell(mR) = -\frac{1}{2}\ln(2\pi) - \frac{1}{4}\ln((mR)^2+\ell^2) + ((mR)^2+\ell^2)^{1/2} - \ell \ln \frac{\ell+\sqrt{\ell^2+(mR)^2}}{mR} + O([(mR)^2+\ell^2]^{-1/2})$$

Therefore

$$\frac{I'_\ell(mR)}{I_\ell(mR)} = -\frac{1}{2}\frac{mR}{(mR)^2+\ell^2} + \frac{mR}{\sqrt{(mR)^2+\ell^2}} + \frac{\ell}{mR} - \frac{\ell mR}{[(mR)^2+\ell^2]^{1/2}[\ell+\sqrt{(mR)^2+\ell^2}]} + O(((mR)^2+\ell^2)^{-1/2}) \tag{3.28}$$

Using expansion (3.28) and using Euler–MacLaurin formula for the sum over ℓ

$$\sum_{\ell=0}^{\ell_{\max}} f(\ell) = \int_0^{\ell_{\max}} f(x)\,dx + \frac{1}{2}[f(0)+f(\ell_{\max})] + \cdots \tag{3.29}$$

we finally find

$$S = mR + \frac{mR}{2}\ln\frac{2}{m\sigma} + O(1) \tag{3.30}$$

We notice that the divergent (when the cutoff $\sigma \to 0$) surface contribution from $\beta\Omega_{\text{cond}}^{\text{ideal}}(m)$ is canceled by the contribution from S giving for the surface tension the already known[3, 9] finite expression

$$\beta\gamma_d = \frac{m}{4} \tag{3.31}$$

Putting all terms together

$$\beta\Omega = -\beta p_b \pi R^2 + \beta\gamma_d 2\pi R + \tfrac{1}{6}\ln(mR) + O(1) \tag{3.32}$$

The grand potential has the expected $(1/6)\ln(mR)$ finite-size correction.

The one-component plasma

As a complement to the above study of the finite-size corrections, in this section we consider another solvable model of Coulomb system, the two-dimensional one-component plasma at $\Gamma = 2$.[14, 15] This systems is composed of N particles with charge q living in a neutralizing uniform background. The one-component plasma in a disk with ideal dielectric boundaries was solved by Smith.[11] The canonical partition function reads

$$Z = \left(\frac{\pi R a}{\lambda_{\text{th}}^2}\right)^N e^{3N^2/4} N^{-N(N+1)/2} \times \prod_{l=1}^{N} \left(\gamma(l, N) - N^{-(2N+1-2l)} \gamma(2N+1-l, N)\right) \tag{3.33}$$

with $\gamma(s, x) = \int_0^x t^{s-1} e^{-t}\, dt$ the incomplete gamma function and $\lambda_{\text{th}} = h/\sqrt{2\pi m k_B T}$ is the thermal wavelength of the particles.

We want to study the large-R expansion of the free energy of the one-component plasma. The free energy can be written as

$$\beta F = \beta F_{\text{hard}} - \sum_{n=0}^{N-1} \ln\left[1 - N^{-(2N-1-2n)} \frac{\gamma(2N-n, n)}{\gamma(1+n, N)}\right] \tag{3.34}$$

where

$$\beta F_{\text{hard}} = -3N^2/4 - N \ln\left(\frac{\pi R a}{\lambda_{\text{th}}^2}\right) + (N(N+1)/2) \ln N - \sum_{n=1}^{N} \ln \gamma(n, N) \tag{3.35}$$

is the free energy of a one-component plasma in a disk with hard walls boundaries.[14] The finite-size expansion of this terms is[7]

$$\beta F_{\text{hard}} = \beta f \pi R^2 + \beta \gamma_{\text{hard}}^{\text{ocp}} 2\pi R + \tfrac{1}{6} \ln\left[(\pi n)^{1/2} R\right] + O(1) \tag{3.36}$$

with the bulk free energy per unit "volume" (surface)

$$\beta f = \frac{n}{2} \ln\left[\frac{n \lambda_{\text{th}}^4}{2\pi^2 a^2}\right] \tag{3.37}$$

and the "surface" (perimeter) contribution

$$\beta\gamma_{\substack{\text{ocp}\\ \text{hard}}} = -\sqrt{\frac{n}{2\pi}} \int_0^\infty \ln \frac{1+\text{erf}(y)}{2}\, dy \tag{3.38}$$

$n = N/(\pi R^2)$ is the density and $\text{erf}(y) = (2/\sqrt{\pi})\int_0^y e^{-x^2}\, dx$ is the error function.

The expansion of the remaining term in equation (3.34) can be obtained with the following uniform asymptotic expansions for the incomplete gamma function

$$\gamma(n+1, N) = \frac{n!}{2}\left[1+\text{erf}\left(\frac{N-n}{\sqrt{2N}}\right) + O(1/\sqrt{N})\right] \tag{3.39a}$$

$$\gamma(2N-n, N) = \frac{(2N-n-1)!}{2} \times\left[1+\text{erf}\left(\frac{n-N}{\sqrt{2N}}\right) + O(1/\sqrt{N})\right] \tag{3.39b}$$

These expansions are valid when $N-n$ is of order $\sqrt{N}$ and the corresponding terms in the sum (3.34) are the ones that give a relevant contribution to βF. Also for n such that $N-n$ is of order $\sqrt{N}$ using Stirling formula we have the expansion for the factorials

$$(2N-n-1)! = N!\, N^{N-n-1}[1+O(1/\sqrt{N})] \tag{3.40a}$$

$$n! = N!\, N^{n-N}[1+O(1/\sqrt{N})] \tag{3.40b}$$

Finally replacing the sum in equation (3.34) by an integral we find

$$\beta F = \beta F_{\text{hard}} - \sqrt{2N} \int_0^\infty \ln \frac{2\,\text{erf}(y)}{1+\text{erf}(y)}\, dy + O(1) \tag{3.41}$$

Putting this last result together with the expansion (3.36) for the hard wall case we find

$$\beta F = \beta f \pi R^2 + \beta\gamma_{\substack{\text{ocp}\\ \text{diel}}} 2\pi R + \tfrac{1}{6} \ln[(\pi n)^{1/2} R] + O(1) \tag{3.42}$$

with

$$\beta\gamma_{\substack{\text{ocp}\\ \text{diel}}} = -\sqrt{\frac{n}{2\pi}} \int_0^\infty \ln \text{erf}(y)\, dy \tag{3.43}$$

For this model we find again the expected universal logarithmic finite-size correction for the free energy.

4. DENSITY AND CORRELATIONS

4.1. Green functions

We return to the study of the two-component plasma. We are now interested in the density and correlations functions. These can be obtained with the Green function

$$\mathbf{G} = \frac{1}{2\pi a \zeta} \frac{\mathbf{K}}{1+\mathbf{K}} \tag{4.1}$$

as explained in ref. 9. The density $n_s(\mathbf{r})$ of particles of sign s is

$$n_s(\mathbf{r}) = \frac{m}{2} \sum_{\alpha} G_{ss}^{\alpha\alpha}(\mathbf{r}, \mathbf{r}) \tag{4.2}$$

and the truncated two-body density is

$$n_{s_1 s_2}^{(2)\,T}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m^2}{2} \sum_{\alpha_1 \alpha_2} G_{s_1 s_2}^{\alpha_1 \alpha_2}(\mathbf{r}_1, \mathbf{r}_2)\, G_{s_2 s_1}^{\alpha_2 \alpha_1}(\mathbf{r}_2, \mathbf{r}_1) \tag{4.3}$$

From its definition (4.1) the Green functions obey the integral equations

$$G_{ss}(\mathbf{r}_1, \mathbf{r}_2) + \frac{m}{2\pi} \int d\mathbf{r} \begin{pmatrix} \dfrac{R}{R^2 - \bar{z}_1 z} & \dfrac{1}{\bar{z} - \bar{z}_1} \\ \dfrac{1}{z - z_1} & \dfrac{R}{R^2 - z_1 \bar{z}} \end{pmatrix} G_{-ss}(\mathbf{r}, \mathbf{r}_2) = 0 \tag{4.4a}$$

$$G_{-ss}(\mathbf{r}_1, \mathbf{r}_2) + \frac{m}{2\pi} \int d\mathbf{r} \begin{pmatrix} \dfrac{R}{R^2 - \bar{z}_1 z} & \dfrac{1}{\bar{z} - \bar{z}_1} \\ \dfrac{1}{z - z_1} & \dfrac{R}{R^2 - z_1 \bar{z}} \end{pmatrix} G_{ss}(\mathbf{r}, \mathbf{r}_2) = \frac{1}{2\pi} \begin{pmatrix} \dfrac{R}{R^2 - \bar{z}_1 z_2} & \dfrac{1}{\bar{z}_2 - \bar{z}_1} \\ \dfrac{1}{z_2 - z_1} & \dfrac{R}{R^2 - z_1 \bar{z}_2} \end{pmatrix} \tag{4.4b}$$

These integral equations can be transformed into the differential equations

$$G_{-ss}(\mathbf{r}_1, \mathbf{r}_2) - \frac{2}{m} \begin{pmatrix} 0 & \partial_{\bar{z}_1} \\ \partial_{z_1} & 0 \end{pmatrix} G_{ss}(\mathbf{r}_1, \mathbf{r}_2) = 0 \tag{4.5a}$$

$$\begin{pmatrix} 0 & \partial_{\bar{z}_1} \\ \partial_{z_1} & 0 \end{pmatrix} G_{-ss}(\mathbf{r}_1, \mathbf{r}_2) - \frac{m}{2} G_{ss}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{2} \delta(\mathbf{r}_1 - \mathbf{r}_2)\, \mathbb{I} \tag{4.5b}$$

where $\mathbb{I}$ is the 2×2 unit matrix. These equations can be combined into

$$\Delta_{\mathbf{r}_1} G_{ss}(\mathbf{r}_1, \mathbf{r}_2) - m^2 G_{ss}(\mathbf{r}_1, \mathbf{r}_2) = -m\delta(\mathbf{r}_1 - \mathbf{r}_2)\, \mathbb{I} \tag{4.6}$$

The boundary conditions can be obtained from the integral equations (4.4). If $\mathbf{r}_1 = \mathbf{R}$, $z_1 = Re^{i\phi_1}$, is on the boundary then from the integral equations (4.4) it can be seen that

$$G_{ss'}^{11}(\mathbf{R}, \mathbf{r}_2) + e^{i\phi_1} G_{ss'}^{21}(\mathbf{R}, \mathbf{r}_2) = 0 \tag{4.7a}$$

$$G_{ss'}^{12}(\mathbf{R}, \mathbf{r}_2) + e^{i\phi_1} G_{ss'}^{22}(\mathbf{R}, \mathbf{r}_2) = 0 \tag{4.7b}$$

For the present disk geometry we look for a solution of equation (4.6) as a Fourier series in ϕ_1. The solution for G_{ss}^{11} and G_{ss}^{21} can be written as

$$G_{ss}^{11}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} e^{i\ell\phi_1} [m e^{-i\ell\phi_2} I_\ell(mr_<)\, K_\ell(mr_>) + A_\ell I_\ell(mr_1)\, I_\ell(mr_2)] \tag{4.8a}$$

$$G_{ss}^{21}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} e^{i\ell\phi_1} B_\ell I_\ell(mr_1) \tag{4.8b}$$

where K_ℓ is a modified Bessel function of order ℓ, $r_< = \min(r_1, r_2)$, $r_> = \max(r_1, r_2)$ and A_ℓ and B_ℓ are constants (with respect to $\mathbf{r}_1$) of integration that will be determined by the boundary conditions (4.7). From equation (4.5a) we have

$$G_{-ss}^{11}(\mathbf{r}_1, \mathbf{r}_2) = \frac{2}{m} \partial_{\bar{z}_1} G_{ss}^{21}(\mathbf{r}_1, \mathbf{r}_2) \tag{4.9a}$$

$$G_{-ss}^{21}(\mathbf{r}_1, \mathbf{r}_2) = \frac{2}{m} \partial_{z_1} G_{ss}^{11}(\mathbf{r}_1, \mathbf{r}_2) \tag{4.9b}$$

Therefore, using equations (4.8) we have, for $r_1 > r_2$,

$$G^{21}_{-ss}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} e^{i(\ell-1)\phi_1} I_\ell(mr_2)[-me^{-i\ell\phi_2} K_{\ell-1}(mr_1) + A_\ell I_{\ell-1}(mr_1)] \tag{4.10a}$$

$$G^{11}_{ss}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} e^{i\ell\phi_1} B_{\ell-1} I_{\ell-1}(mr_1) \tag{4.10b}$$

Using the boundary conditions (4.7) we find the following linear system of equations for A_ℓ and B_ℓ

$$A_\ell I_\ell(mr_2)\, I_\ell + B_{\ell-1} I_{\ell-1} = -me^{-i\ell\phi_2} K_\ell I_\ell(mr_2) \tag{4.11a}$$

$$A_\ell I_\ell(mr_2)\, I_{\ell-1} + B_{\ell-1} I_\ell = me^{-i\ell\phi_2} K_{\ell-1} I_\ell(mr_2) \tag{4.11b}$$

where $I_\ell = I_\ell(mR)$ and similar definitions for the other Bessel functions without argument. The solution of this linear system is

$$A_\ell = -me^{-i\ell\phi_2} \frac{K_\ell I_\ell + K_{\ell-1} I_{\ell-1}}{I_\ell^2 - I_{\ell-1}^2} \tag{4.12}$$

$$B_{\ell-1} = \frac{e^{-i\ell\phi_2}}{R} \frac{I_\ell(mr_2)}{I_\ell^2 - I_{\ell-1}^2} \tag{4.13}$$

And finally,

$$G^{11}_{ss}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi} K_0(m|\mathbf{r}_1 - \mathbf{r}_2|) + \frac{m}{2\pi} \sum_{\ell \in \mathbb{Z}} \frac{K_\ell I_\ell + K_{\ell-1} I_{\ell-1}}{I_{\ell-1}^2 - I_\ell^2} I_\ell(mr_1)\, I_\ell(mr_2)\, e^{i\ell(\phi_1 - \phi_2)} \tag{4.14a}$$

$$G^{21}_{ss}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi R} \sum_{\ell \in \mathbb{Z}} \frac{I_\ell(mr_1)\, I_{\ell+1}(mr_2)}{I_{\ell+1}^2 - I_\ell^2} e^{i\ell\phi_1 - i(\ell+1)\phi_2} \tag{4.14b}$$

$$G^{11}_{-ss}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi R} \sum_{\ell \in \mathbb{Z}} \frac{I_\ell(mr_1)\, I_\ell(mr_2)}{I_\ell^2 - I_{\ell-1}^2} e^{i\ell(\phi_1 - \phi_2)} \tag{4.14c}$$

$$G^{21}_{-ss}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi} \frac{\bar{z}_2 - \bar{z}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} K_1(m\,|\mathbf{r}_1 - \mathbf{r}_2|) + \frac{m}{2\pi} \sum_{\ell \in \mathbb{Z}} \frac{K_\ell I_\ell + K_{\ell-1} I_{\ell-1}}{I_{\ell-1}^2 - I_\ell^2} I_{\ell-1}(mr_1)\, I_\ell(mr_2)\, e^{i(\ell-1)\phi_1 - i\ell\phi_2} \tag{4.14d}$$

From equations (4.4) it can be seen that the remaining Green functions can be easily deduced since they obey

$$G^{12}_{ss'}(\mathbf{r}_1, \mathbf{r}_2) = \overline{G^{21}_{ss'}(\mathbf{r}_1, \mathbf{r}_2)} \tag{4.15a}$$

$$G^{22}_{ss'}(\mathbf{r}_1, \mathbf{r}_2) = \overline{G^{11}_{ss'}(\mathbf{r}_1, \mathbf{r}_2)} \tag{4.15b}$$

Other useful symmetry relations between the Green functions are

$$G^{11}_{ss'}(\mathbf{r}_1, \mathbf{r}_2) = \overline{G^{11}_{ss'}(\mathbf{r}_2, \mathbf{r}_1)} \qquad G^{22}_{ss'}(\mathbf{r}_1, \mathbf{r}_2) = \overline{G^{22}_{ss'}(\mathbf{r}_2, \mathbf{r}_1)} \tag{4.15c}$$

$$G^{21}_{ss'}(\mathbf{r}_1, \mathbf{r}_2) = -G^{21}_{ss'}(\mathbf{r}_2, \mathbf{r}_1) \qquad G^{12}_{ss'}(\mathbf{r}_1, \mathbf{r}_2) = -G^{12}_{ss'}(\mathbf{r}_2, \mathbf{r}_1) \tag{4.15d}$$

4.2. Density

The density is obtained from equation (4.2) and it reads

$$n_s(\mathbf{r}) = n_b + \frac{m^2}{2\pi} \sum_{\ell \in \mathbb{Z}} \frac{K_\ell I_\ell + I_{\ell-1} K_{\ell-1}}{I_\ell^2 - I_{\ell-1}^2} I_\ell(mr)^2 \tag{4.16}$$

where n_b is the bulk density of the infinite system which is formally divergent when the cutoff vanishes. Writing formally the bulk density as

$$n_b = \frac{m}{2\pi} \sum_{\ell \in \mathbb{Z}} I_\ell K_\ell, \tag{4.17}$$

rearranging the terms in equation (4.16) and using the Wronskian

$$I_\ell K_{\ell-1} + I_{\ell-1} K_\ell = \frac{1}{mR} \tag{4.18}$$

the density at the boundary can be written as

$$n_s(R) = \frac{1}{2\pi R} \sum_{\ell \in \mathbb{Z}} \frac{I_\ell I_{\ell-1}}{I_{\ell-1}^2 I_\ell^2} \tag{4.19}$$

The above expression clearly vanishes since the term ℓ is canceled by the term $-\ell+1$. So we recover the expected result

$$n_s(R) = 0 \tag{4.20}$$

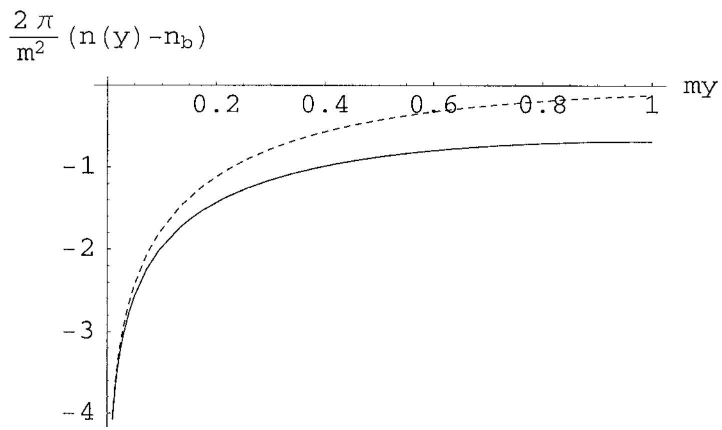

Fig. 1. Difference between the charge density and the bulk charge density $n_s(y)-n_b$ as a function of the distance $y=R-r$ from the wall. The dashed curve represents case of a two-component plasma near an infinite plane wall. The solid curve represents the density in the disk case with $mR=1$.

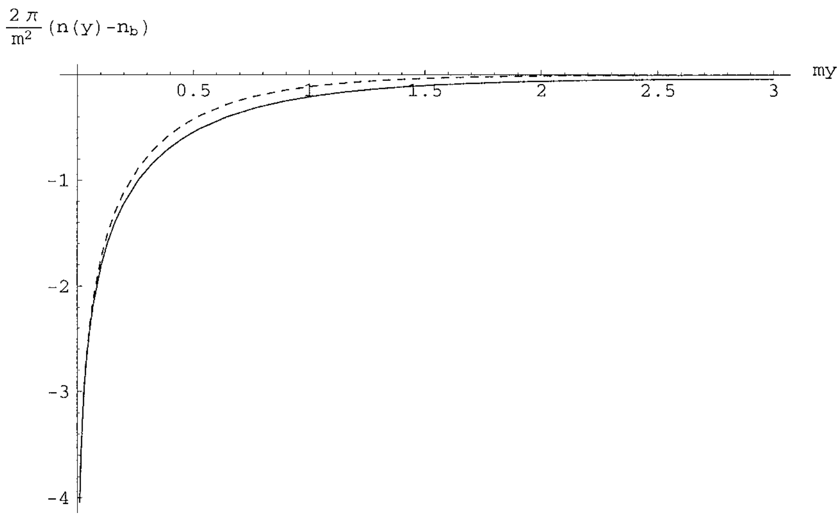

Fig. 2. Difference between the charge density and the bulk charge density $n_s(y)-n_b$ as a function of the distance $y=R-r$ from the wall. The dashed curve represents case of a two-component plasma near an infinite plane wall. The solid curve represents the density in the disk case with $mR=3$.

The strong repulsion between a charge and its image cause the density at the boundary to vanish.

In ref. 9 it was shown that the density near an infinite ideal dielectric plane wall is

$$n_s(y) = n_b - \frac{m^2}{2\pi} K_0(2my) \tag{4.21}$$

where y is the distance from the wall. To compare our result for the density in a disk and result (4.21) near an infinite plane wall we plot in Figs. 1 and 2 both densities as a function of the distance from the wall for disks of different sizes $R = 1/m$ and $R = 3/m$. In both cases the density for small distances y behaves as

$$n_s(y) - n_b = \frac{m^2}{2\pi} \ln(2my) + O(1) \tag{4.22}$$

It can also be seen in Figs. 1 and 2 that the density decays faster for the semi-infinite system (plane wall) than in the disk case. This effect is stronger for the small disk $mR = 1$.

4.3. Correlations

From the Green functions (4.14) we obtain the two-body correlation functions using equation (4.3). Using the symmetry relations (4.15) the correlation between a particle of sign s at $\mathbf{r}_1$ and a particle of sign s' at $\mathbf{r}_2$ reads

$$n_{ss'}^{(2)\,T}(\mathbf{r}_1, \mathbf{r}_2) = -m^2[|G_{ss'}^{11}(\mathbf{r}_1, \mathbf{r}_2)|^2 - |G_{ss'}^{21}(\mathbf{r}_1, \mathbf{r}_2)|^2] \tag{4.23}$$

This gives

$$\begin{aligned} n_{ss}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m^4}{(2\pi)^2} \Bigg| & K_0(m\,|\mathbf{r}_1 - \mathbf{r}_2|) \\ & + \sum_{\ell \in \mathbb{Z}} \frac{K_\ell I_\ell + K_{\ell-1} I_{\ell-1}}{I_{\ell-1}^2 - I_\ell^2} I_\ell(mr_1)\, I_\ell(mr_2)\, e^{i\ell(\phi_1 - \phi_2)} \Bigg|^2 \\ & + \left(\frac{m}{2\pi R}\right)^2 \left| \sum_{\ell \in \mathbb{Z}} \frac{I_\ell(mr_1) I_{\ell+1}(mr_2)}{I_{\ell+1}^2 - I_\ell^2}\, e^{i\ell\phi_1 - i(\ell+1)\phi_2} \right|^2 \end{aligned} \tag{4.24a}$$

and

$$n_{-ss}^{(2)\,T}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m^4}{(2\pi)^2} \left| \frac{\bar{z}_2 - \bar{z}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} K_1(m\,|\mathbf{r}_1 - \mathbf{r}_2|) + \sum_{\ell \in \mathbb{Z}} \frac{K_\ell I_\ell + K_{\ell-1} I_{\ell-1}}{I_{\ell-1}^2 - I_\ell^2} I_{\ell-1}(mr_1)\, I_\ell(mr_2)\, e^{i(\ell-1)\,\phi_1 - i\ell\phi_2} \right|^2 - \left(\frac{m}{2\pi R}\right)^2 \left| \sum_{\ell \in \mathbb{Z}} \frac{I_\ell(mr_1) I_\ell(mr_2)}{I_\ell^2 - I_{\ell-1}^2} e^{i\ell(\phi_1 - \phi_2)} \right|^2 \tag{4.24b}$$

From equation (4.23) and the boundary conditions (4.7) it is clear that if one point is on the boundary

$$n_{ss'}^{(2)\,T}(\mathbf{r}_1, \mathbf{r}_2) = 0 \qquad \text{if} \quad \mathbf{r}_1 \in \partial D \text{ or } \mathbf{r}_2 \in \partial D \tag{4.25}$$

as expected due to the strong repulsion between a charge and its image.

If $\mathbf{r}_2 = 0$ the above expressions (4.24) simplify to

$$n_{ss}^{(2)T}(\mathbf{r}, 0) = -\left(\frac{m^2}{2\pi}\right)^2 \left[K_0(mr) + \frac{K_0 I_0 + K_1 I_1}{I_1^2 - I_0^2} I_0(mr) \right]^2 + \left(\frac{m}{2\pi R}\right)^2 \frac{I_1(mr)^2}{(I_1^2 - I_0^2)^2} \tag{4.26a}$$

$$n_{-ss}^{(2)T}(\mathbf{r}, 0) = \left(\frac{m^2}{2\pi}\right)^2 \left[-K_1(mr) + \frac{K_0 I_0 + K_1 I_1}{I_1^2 - I_0^2} I_1(mr) \right]^2 - \left(\frac{m}{2\pi R}\right)^2 \frac{I_0(mr)^2}{(I_1^2 - I_0^2)^2} \tag{4.26b}$$

It is interesting to compare these expressions with the bulk correlations for an infinite system[5]

$$n_{ss,\,\text{bulk}}^{(2)\,T}(\mathbf{r}) = -\left(\frac{m^2}{2\pi}\right)^2 [K_0(mr)]^2 \tag{4.27a}$$

$$n_{-ss,\,\text{bulk}}^{(2)\,T}(\mathbf{r}) = \left(\frac{m^2}{2\pi}\right)^2 [K_1(mr)]^2 \tag{4.27b}$$

Figures 3 and 4 show the two-body density $n_{ss}^{(2)T}(\mathbf{r}, 0)$ for particles of same sign compared to the bulk values for different values of R and Figs. 5 and 6 show the two-body density $n_{-ss}^{(2)T}(\mathbf{r}, 0)$ for particles of different sign.

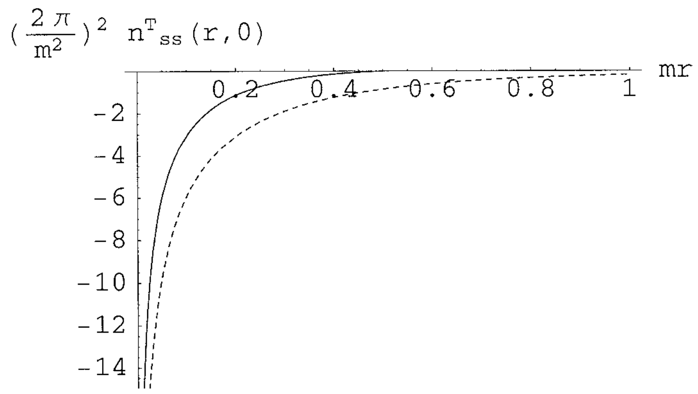

Fig. 3. Two-body density $n_{ss}^{(2)T}(\mathbf{r}, 0)$ with one point fixed on the center of the disk for $mR = 1$. The dashed curve represents the bulk correlation and the solid curve the correlation for the disk case.

For a small disk with $mR = 1$ there is a notable difference. In the disk case the correlations decay faster than in the bulk. This can be easily understood since there is a strong repulsion between a particle and the boundary. But this difference can be hardly noted if the disk is larger. For $mR = 3$ it can be seen in Fig. 4 that the difference between the bulk and the disk case is very small (notice the change of scale in the vertical axis between Figs. 3 and 4).

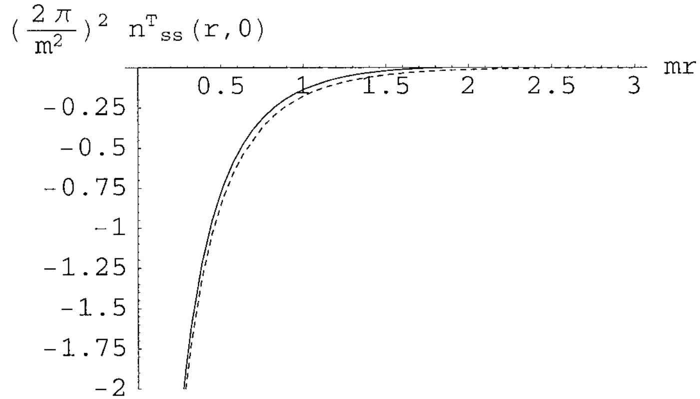

Fig. 4. Two-body density $n_{ss}^{(2)T}(\mathbf{r}, 0)$ with one point fixed on the center of the disk for $mR = 3$. The dashed curve represents the bulk correlation and the solid curve the correlation for the disk case.

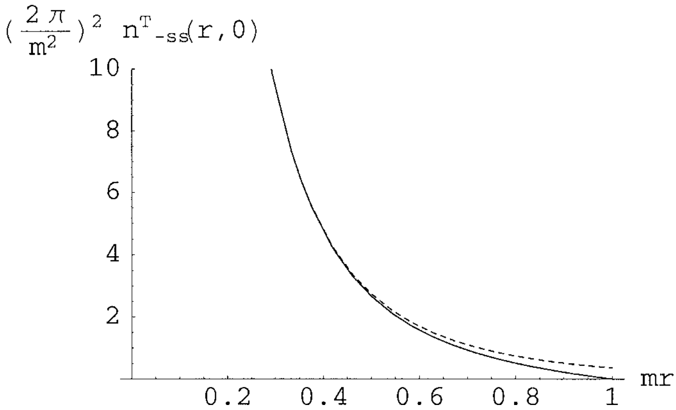

Fig. 5. Two-body density $n^{(2)T}_{-ss}(\mathbf{r}, 0)$ with one point fixed on the center of the disk for $mR = 1$. The dashed curve represents the bulk correlation and the solid curve the correlation for the disk case.

In Figs. 7 and 8 we plot the structure function (charge-charge correlation)

$$S(\mathbf{r}_1, \mathbf{r}_2) = 2(n^{(2)T}_{ss}(\mathbf{r}_1, \mathbf{r}_2) - n^{(2)T}_{-ss}(\mathbf{r}_1, \mathbf{r}_2)) \tag{4.28}$$

with one point fixed at center of the disk $\mathbf{r}_2 = 0$. For the small disk $mR = 1$ there is a clear difference between the bulk case and the disk case. Due to the repulsion between a particle and its image, the screening cloud is more

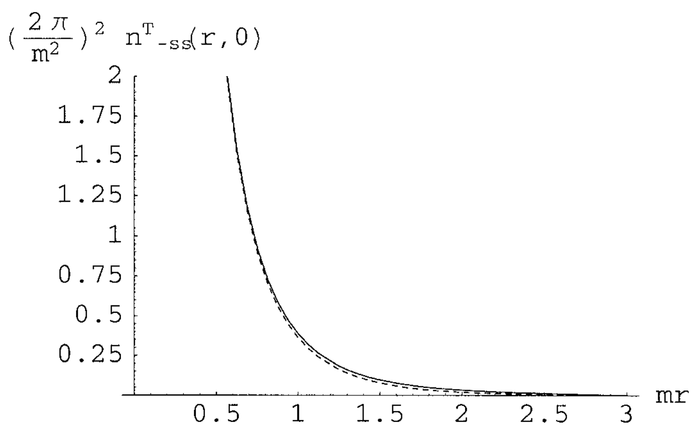

Fig. 6. Two-body density $n^{(2)T}_{-ss}(\mathbf{r}, 0)$ with one point fixed on the center of the disk for $mR = 3$. The dashed curve represents the bulk correlation and the solid curve the correlation for the disk case.

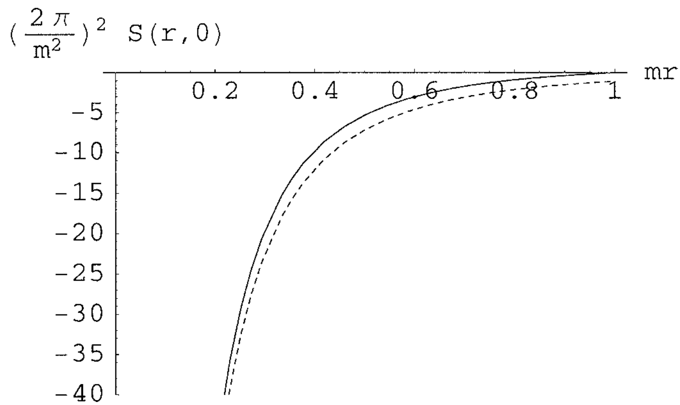

Fig. 7. Structure function $S(\mathbf{r}, 0)$ with one point fixed on the center of the disk for $mR = 1$. The dashed curve represents the bulk correlation and the solid curve the correlation for the disk case.

concentrated in the center of the disk than in the bulk case. But for the large disk $mR = 3$ the difference is hardly noticeable. Notice again the change of scale between Figs. 7 and 8. It is interesting to note that there is not much difference between the bulk and the disk case if the radius of the disk is a few orders the screening length m^{-1} and larger. We only notice differences when $R \sim m^{-1}$ and smaller.

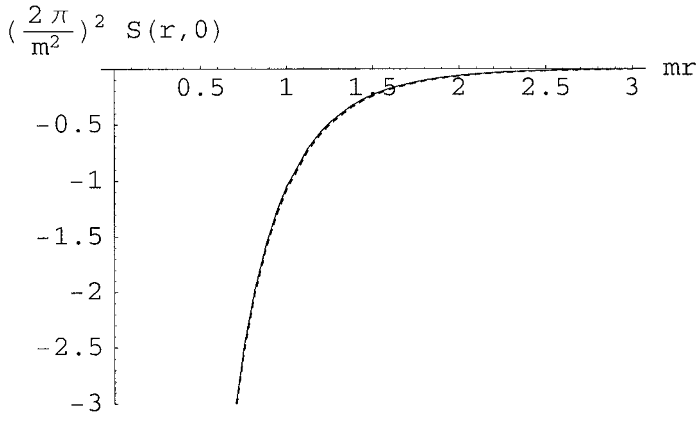

Fig. 8. Structure function $S(\mathbf{r}, 0)$ with one point fixed on the center of the disk for $mR = 3$. The dashed curve represents the bulk correlation and the solid curve the correlation for the disk case.

This can actually be understood analytically. The large-R expression of the correlation functions (4.26) reads

$$n_{ss}^{(2)T}(\mathbf{r}, 0) = \frac{m^4}{(2\pi)^2}\left[-K_0(mr)^2 + 4\pi mR\, e^{-2mR} K_0(mr)\, I_0(mr)\right] + o(mR\, e^{-2mR}) \tag{4.29a}$$

$$n_{-ss}^{(2)T}(\mathbf{r}, 0) = \frac{m^4}{(2\pi)^2}\left[K_1(mr)^2 + 4\pi mR\, e^{-2mR} K_1(mr)\, I_1(mr)\right] + o(mR\, e^{-2mR}) \tag{4.29b}$$

The first correction to the bulk values of the correlation functions is exponentially small for large R.

5. CONCLUSION

We have studied the two-component plasma with coupling constant $\Gamma = 2$ confined in a disk of radius R with ideal dielectric boundaries: the electric potential obeys Neumann boundary conditions. The model is solvable by using a mapping of the Coulomb system onto a four-component free Fermi field. We have computed the grand potential, the density and correlation functions.

The grand potential can be formally written as an average of the grand potential for ideal conductor boundaries and the same grand potential for ideal conductor boundaries but with the sign of the fugacity changed. For ideal conductor boundaries the surface tension is infinite when the cutoff vanishes. Here, the average makes the surface tension finite. This fact also appears for a two-component plasma in a strip.[9]

The Neumann boundary conditions for the electric potential being conformally invariant it is expected that the grand potential of the system exhibits a universal finite-size correction $(1/6)\ln R$. This was explicitly checked on this solvable model. We also checked this universal finite-size correction on the model of the one-component plasma at $\Gamma = 2$ in the same confined geometry which was solved some time ago.[11] In this case is the free energy which exhibits the universal logarithmic finite-size correction.

The density vanishes for a point on the boundary of the disk. This is expected since there is a strong repulsion between the particles and the boundary due to the image forces. This is also true for the correlations, they vanish if one point is on the boundary. We compared the correlations functions for an infinite system without boundaries and the present system in a disk. Due to the repulsion between the particles and the boundary, the screening cloud around a particle in the center of the disk is smaller than the one for an infinite system. But the difference between the screening clouds is only noticeable for disks with radius of the order of the screening length and smaller. If the disk has a radius a few orders larger than the screening length, the difference can be hardly noted. Actually the corrections to the bulk values of the correlations functions are exponentially small for large disks.

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